Electrodynamics in Noncommutative Curved Space Time

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Abstract. We study the issue of the electrodynamics theory in noncommutative curved space time (NCCST) with a new star-product. In this paper, the motion equation of electrodynamics and canonical energy-momentum tensor in noncommutative curved space time will be found. The most important point is the assumption of the noncommutative parameter (θ) be x^μ-independent.

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Introduction

Since Newton the concept of space time has gone through various changes. All stages, however, had in common the notion of a continues linear space. Today we formulate fundamental laws of physics, field theories, gauge field theories, and the theory of gravity on differentiable manifolds. That a change in the concept of space for very short distances might be necessary was already anticipated by Riemann. There are indications today that at very short distances we might have to go beyond differential manifolds. This is only one of several arguments that we have to expect some changes in physics for very short distances. Other arguments are based on the singularity problem in quantum field theory and the fact theory of gravity is non re-normalizable when quantized. Why not try an algebraic concept of space time that could guide us to changes in our present formulation of laws of physics? This is different from the discovery of quantum mechanics. There physics data forced us to introduce the concept of noncommutativity.

There are many approaches to noncommutative geometry and its use in physics. The operator algebra and $\mathcal{C}^{\star}$-algebra one, the deformation quantization one, the quantum group one, and the matrix algebra (fuzzy geometry) one. These show that the concepts of noncommutative subjects were born many years ago, where the idea of noncommutative coordinates is almost as old as quantum mechanics.

Many people are working in the noncommutative field theory to find the details of it. It is a nonlocal theory and for this reason, the NCFT has many ambiguous problems such as UV/IR divergence and causality. The UV/IR problem arises of non-locality directly and the causality is broken when $\theta^{0i} \neq 0$ so as a field theory it is not appealing [1, 2]. In addition, some people thought that the noncommutative field theory violets the Lorentz invariance, but a group of physicists have shown that the noncommutative field theory holds the Lorentz invariance by Hopf algebra [2]. With all these issues, many of physicists prefer to work in this field, because they think it is one of the best candidate for the brighter future of physics and they hope that these problems would be solved in futures.

The formulation of the classical and quantum field theories in NCCST has been a very active subject over the last few years. Most of these approaches focus on free or interacting QFTs on the Moyal-Weyl deformed or κ-deformed Minkowski space time. But at now, many people are working on the quantum field theory and related them on the noncommutative curved space time so this work is one of them. Finally, we hope that these works might solve some conceptual problems of physics that are

still left at very short distances [1, 2, 3, 4].

We introduce new star-product briefly. By deforming the ordinary Moyal $\star$-product, we propose a new Moyal-Weyl $\triangleright$-product which takes into consideration the missing terms cited above which generate gravitational terms to the order θ^2. In reference [5], the authors have shown that we can go to curved space time with replacing of operators variables. We can replace the noncommutative flat space time coordinates variables $[\hat{x}^\mu\ ,\ \hat{x}^\nu] = \imath\theta^{\mu\nu}$ with noncommutative curved space time coordinates variables $\hat{X}^\mu = \hat{x}^\mu + \imath^2 \frac{\theta^{ab}\theta^{\alpha\beta}}{2\sqrt{-\hat{g}}}\partial_b \hat{R}^\mu_{a\alpha\beta}$ where $\hat{R}^\mu_{b\alpha\beta}$ stands for Riemann curvature tensor and $\sqrt{-\hat{g}}$ is determinant of metric as two functions in noncommutative coordinates. We know $[\hat{X}^\mu\ ,\ \hat{X}^\nu] = \imath\theta^{\mu\nu}$ so for two any smooth function A and B we have

$$\hat{A}(\hat{X})\hat{B}(\hat{X}) = A(x)\ e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \otimes \overrightarrow{\partial}_\nu + \imath(\imath^2 \frac{\theta^{ab}\theta^{\alpha\beta}}{2\sqrt{-g}}\partial_b R^\mu_{a\alpha\beta})\partial_\mu\ (\Im_A \otimes \Im_B)}\ B(x)$$

where $\Im$ stands for the identity of spaces (for example $\Im_A$ is identity of "A"). If $\triangle^\mu = \imath^2 \frac{\theta^{ab}\theta^{\alpha\beta}}{2\sqrt{-g}}\partial_b \hat{R}^\mu_{a\alpha\beta}$ and $\triangleright \equiv e^{\triangle x^\mu \partial_\mu (\Im\otimes\Im) + \frac{\imath\theta^{\mu\nu}}{2}\partial_\mu \otimes \partial_\nu}$ so we have $(A \triangleright B)(\hat{x}) = A\ e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu \otimes \overrightarrow{\partial}_\nu + \imath\triangle^\mu \partial_\mu\ (\Im_A \otimes \Im_B)}\ B$. If the $\triangle$ is x^μ-independent we can write

$$\begin{aligned}
\int d^{(d-1)}x\ (A \triangleright B)(\hat{x}) &= \int d^{(d-1)}x \int \mathbf{dkdq}\tilde{A}(k)\tilde{B}(q)\ e^{\imath(k+q)\cdot\hat{x} - \frac{\imath}{2}\theta^{\mu\nu}k_\mu p_\nu + \imath\triangle^\mu (k+q)_\mu} \\
&= \int d^{(d-1)}x\ e^{\triangle^\mu \partial_\mu} \int \mathbf{dkdq}\tilde{A}(k)\tilde{B}(q)\ e^{\imath(k+q)\cdot\hat{x} - \frac{\imath}{2}\theta^{\mu\nu}k_\mu p_\nu} \\
&= \int d^{(d-1)}x (A(x) \star B(x) + \triangle^\mu \partial_\mu (A(x)B(x)) + 0(\theta^4)) \\
&= \int d^{(d-1)}x A(x) \star B(x) \qquad (1)
\end{aligned}$$

where $\mathbf{dq} = \frac{1}{(2\pi)^{\frac{d-1}{2}}}dq$ and we can remove the total derivative term when $\triangle$ be a constant. At last, there is a quality below of integral between of old and new star-product for some manifolds with special metric which $\triangle$ over of them will be a constant.

We introduce a new symbol $S_\star(A_1, A_2, .., A_n)$ that it includes all symmetric compounds of A_1-A_n. As a function in integration, because of rotational properties the one of A_i from $S_\star(A_1, A_2, .., A_n)$ can be written in the first (last) of all compounds of $S_\star$ or $S_\star(A_i; A_{i-1}, .., A_n, ..)$ where A_i is a selected function. Also, we can show that $\star$-product in integral formalism has following property

$$\int d^d x\ S_\star(A_1, A_2, .., A_n) = \int d^d x\ S_\star(A_i; A_{i-1}, .., A_n, ..) = \int d^d x\ A_i \star \bar{S}^i_\star(A_{i-1}, .., A_n, ..)$$

where $\bar{S}^i_\star$ means $S_\star$ without A_i and includes all symmetric compounds of remanned $A_1, .., A_{i-1}, A_{i+1}, .., A_n$. The $S_\star(A_1, .., A_n)$ has $n!$-terms out of integration but in the integration formalism it reduces to n-terms. Suppose, that A-function has appeared twice in $S_\star(B, C, .., A, D, .., A, G, .., H)$, so for $I = \int d^dx\, S_\star(B, C, .., A, D, .., A, G, .., H)$ from Leibnitz rule we have

$$\begin{aligned}
\frac{\delta}{\delta A(z)} I &= \int d^dx\ S_\star(B, C, .., \frac{\delta}{\delta A(z)} A, D, .., A, G, .., H) \\
&+ S_\star(B, C, .., A, D, .., \frac{\delta}{\delta A(z)} A, G, .., H) \\
&= \int d^dx\ \frac{\delta}{\delta A(z)} A \star \bar{S}^1_\star(D, .., A, G, .., H, B, C, ..) \\
&+ \int d^dx\ \frac{\delta}{\delta A(z)} A \star \bar{S}^2_\star(G, .., H, B, C, .., A, D, ..) \\
&= \int d^dx\ \frac{\delta}{\delta A(z)} S^1_\star(A; D, .., \acute{A}, G, .., H, B, C, ..) \\
&+ \int d^dx\ \frac{\delta}{\delta A(z)} S^2_\star(A; G, .., H, B, C, .., \acute{A}, D, ..) \\
&= 2\bar{S}^{1\ or\ 2}_\star
\end{aligned} \tag{2}$$

Physics Notes and the Motion Equation of Fields

We start by showing how to construct an action electrodynamics in NCCST. If one direct follows the general rule of transforming usual theories in noncommutative ones by replacing product of fields by star product [2, 3] and we believe that this change should be done on Lagrangian density. In fact, $\mathbf{S}_{Cm}(\mathcal{L}_{Cm}) \to \mathbf{S}_{Nc}(\mathcal{L}^{Sym}_{Nc})$ or

$$\mathbf{S} = \int d^dx\ \sqrt{-g} \triangleright \mathcal{L}^{\ Sym}_{\triangleright Nc}$$

where $\triangleright$ is a new star product. The classical Lagrangian density is

$$\mathcal{L} = \frac{-1}{4}(F_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}) \tag{3}$$

where $g^{\mu\nu}$,$-g$ and $F^{\mu\nu}$ are the metric tensor, the determinant of metric and the strength fields $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - \imath\ e[\ A^\mu, A^\nu\ ]_\star$ respectively. The building of action in NCCST is more complicated because symmetric ordering must also considered. In particular interested in the deformation of the canonical action goes to

$$\mathbf{S} \ = \ \int d^dx\ \ \frac{-1}{4}\sqrt{-g} \triangleright S_\triangleright(F_{\alpha\beta}, g^{\alpha\mu}, g^{\beta\nu}, F_{\mu\nu}) \tag{4}$$

As for multiplication of the functions we can write $\int (f \triangleright g)(x) = \int (f \star g)(\bar{x})$ where $\bar{x}^\mu = x^\mu + \triangle^\mu$ and if $\bar{f} = f(\bar{x})$ so then $(f \star g)(\bar{x}) = \bar{f} \star \bar{g}$. The earlier star product ($\star$-product) will be an associative by the Drinfel'd map. By using these tools, and hermitical structure we have

$$S_\triangleright(F_{\alpha\beta}, g^{\alpha\mu}, g^{\beta\nu}, F_{\mu\nu}) = S_\star(\bar{F}_{\alpha\beta}, \bar{g}^{\alpha\mu}, \bar{g}^{\beta\nu}, \bar{F}_{\mu\nu}) = \bar{F} \star \bar{g} \star \bar{g} \star \bar{F} \tag{5}$$

but we work with $\mathcal{L}_\star = S_\star(F, g, g, F)$. Now, we can deform the classical action to the global expression

$$\mathbf{S} = \int d^dx \ \sqrt{-\bar{g}} \star \mathcal{L}_\star = \int d^dx \ \sqrt{-g} \star \mathcal{L}_\star + total \ \ derivative \tag{6}$$

but when the $\triangle$ be x-independent the last term is zero and also one of $\star$'s can be removed $\int f \star g \star h = \int h \star g \star f$. While the space time without tersion we can use from $F^{\mu\nu}$. For research of the motion equation of fields we start from the principle of the least action $\frac{\delta \mathbf{S}}{\delta A^\kappa(z)} = 0$, namely,

$$\begin{aligned} \frac{\delta \mathbf{S}}{\delta A^\kappa(z)} &= \frac{\delta}{\delta A^\kappa(z)} \int d^dx \ \frac{-\sqrt{-g}}{4} \star S_\star(F, g, g, F) \\ &= \frac{\delta}{\delta A^\kappa(z)} \int d^dx \ \frac{-1}{4} S_\star(\sqrt{-g}; F, g, g, F) \\ &= \frac{\delta}{\delta A^\kappa(z)} \int d^dx \ \frac{-1}{2} S_\star(F; \sqrt{-g}, F, g, g) \end{aligned} \tag{7}$$

so we have

$$\partial_\mu S_\star(\sqrt{-g}, F_{\alpha\beta}, g^{\kappa\alpha}, g^{\mu\beta}) - \imath e[A_\mu, \ S_\star(\sqrt{-g}, F_{\alpha\beta}, g^{\kappa\alpha}, g^{\mu\beta})]_\star = 0 \tag{8}$$

for Lagranian is diffined in Eq.5 we have

$$\partial_\mu \mathcal{F}^{\kappa\mu} - \imath e[A_\mu, \ \mathcal{F}^{\kappa\mu}] = 0 \tag{9}$$

where $\mathcal{F}^{\kappa\mu} = \{g^{\kappa\beta}, \ g^{\mu\alpha}\} \star F_{\alpha\beta} \star \sqrt{-g} + \sqrt{-g} \star F_{\alpha\beta} \star \{g^{\kappa\beta}, \ g^{\mu\alpha}\}$ It is the motion equation of electrodynamics fields in NCCST while the $\triangle$ be a constant.

the Formal Canonical Energy – momentum Tensor

It is useful to study the derivative the different pieces of the action with respect to $g^{\mu\nu}$. In view of the physical interpretations to be given later we introduce the new tensor $\mathbf{T}^{\mu\nu}$ via

$$\frac{\delta \mathbf{S}}{\delta g_{\mu\nu}} = -\frac{1}{2} S_\triangleright(\sqrt{-g} \ , \ \mathbf{T}^{\mu\nu})$$

as the field symmetric energy-momentum tensor. Consider the action of field for $\theta^{0i} \neq 0$. In this case, we can remove all $\star$'s related to metric tensor, because we do not search the momentum of metric [6]. However, for any case of noncommutativity we can start from

$$\mathbf{S}^{\theta^{0i}=0} = \int d^dx\sqrt{-g} \triangleright \mathcal{L}_{\triangleright}^{\theta^{0i}=0}, \quad \mathbf{S}^{\theta^{0i}\neq 0} = \int d^dx\sqrt{-g}\mathcal{L}_{\triangleright}^{\theta^{0i}\neq 0} \tag{10}$$

where $\mathcal{L}_{\star}^{\theta^{0i}=0} = S_{\star}(F, g, g, F)$ and $\mathcal{L}_{\star}^{\theta^{0i}\neq 0} = \frac{-1}{4} g^{\mu\alpha} g^{\nu\beta}(F_{\mu\nu} \star F_{\alpha\beta})$. Variation with respect to $g_{\mu\nu}$ for two cases give

$$\begin{aligned}
\delta\mathbf{S}^{\theta^{0i}=0} &= \int d^dx(\delta\sqrt{-g} \star \mathcal{L}_{\star}^{\theta^{0i}=0} + \sqrt{-g} \star \delta\mathcal{L}_{\star}^{\theta^{0i}=0}) \\
\delta\mathbf{S}^{\theta^{0i}\neq 0} &= \int d^dx(\delta\sqrt{-g}\ \mathcal{L}_{\star}^{\theta^{0i}\neq 0} + \sqrt{-g}\ \delta\mathcal{L}_{\star}^{\theta^{0i}\neq 0})
\end{aligned} \tag{11}$$

We first perform the variation of $\sqrt{-g}$ with respect to $\delta g_{\mu\nu}$ [7]. For this we write $\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g$ and observe that by varying $g_{\mu\nu}(\bar{x})$, the variation of determinant g involves the co-factors, which in fact are equal to g times the inverse, $g^{\mu\nu}$ is $\delta g = -g\ g_{\mu\nu}\delta g^{\mu\nu}$ so we have

$$\delta\ \sqrt{-g} = \frac{1}{2} S_{\star}(\sqrt{-g},\ g^{\mu\nu},\ \delta g_{\mu\nu}) = \frac{-1}{2} S_{\star}(\sqrt{-g},\ g_{\mu\nu},\ \delta g^{\mu\nu}) \tag{12}$$

Therefore, we can write the variation of action

$$\delta\mathbf{S}^{\theta^{0i}\neq 0} = \int d^dx((\frac{-1}{2} S_{\star}(\sqrt{-g}, g_{\mu\nu}, \delta g^{\mu\nu}))\ \mathcal{L}_{\star}^{\theta^{0i}\neq 0} + \sqrt{-g}\ \delta\mathcal{L}_{\star}^{\theta^{0i}\neq 0}) \tag{13}$$

and

$$\delta\mathbf{S}^{\theta^{0i}=0} = \int d^dx((\frac{-1}{2} S_{\star}(\sqrt{-g}, g_{\mu\nu}, \delta g^{\mu\nu})) \star\ \mathcal{L}_{\star}^{\theta^{0i}=0} + \sqrt{-g} \star \delta\mathcal{L}_{\star}^{\theta^{0i}=0}) \tag{14}$$

Consider now the variation of Lagrangian density, first case $\theta^{0i} \neq 0 \Rightarrow \delta\mathcal{L}_{\star}^{\theta^{0i}\neq 0} = \frac{-1}{4}((\delta g^{\mu\alpha}) g^{\nu\beta}(F_{\mu\nu} \star F_{\alpha\beta}) + g^{\mu\alpha}(\delta g^{\nu\beta})(F_{\mu\nu} \star F_{\alpha\beta}))$ and second case $\theta^{oi} = 0 \Rightarrow \delta\mathcal{L}_{\star}^{\theta^{0i}=0} = \frac{-1}{4}(F_{\mu\nu} \star \delta g^{\mu\alpha} \star g^{\nu\beta} \star F_{\alpha\beta} + F_{\mu\nu} \star g^{\mu\alpha} \star \delta g^{\nu\beta} \star F_{\alpha\beta})$ so we get to

$$\begin{aligned}
\mathbf{T}_{\lambda\kappa}^{\theta^{0i}\neq 0} &=: \frac{\partial}{\partial\sqrt{-g}} S_{\star}(\sqrt{-g}, g_{\lambda\kappa}, \mathcal{L}_{\star}^{\theta^{0i}\neq 0}) + g^{\mu\nu} S_{\star}(F_{\mu\kappa},\ F_{\nu\lambda}) \\
\mathbf{T}_{\lambda\kappa}^{\theta^{0i}=0} &= \frac{\partial}{\partial\sqrt{-g}} S_{\star}(\sqrt{-g}, g_{\lambda\kappa}, \mathcal{L}_{\star}^{\theta^{0i}=0}) + \frac{\partial}{\partial\sqrt{-g}} S_{\star}(g^{\nu\beta}, F_{\lambda\nu}, \sqrt{-g}, F_{\kappa\beta})
\end{aligned} \tag{15}$$

and all $\mathbf{T}_{\mu\nu}$ are symmetric tensors.

Discussion

In this work we could drive the action of quantum electrodynamics in noncommutative curved space time. Also we find the motion equation of electrodynamics fields. We also assumed that $\theta^{0i} \neq 0$ so the momentum conjugate of $g^{\mu\nu}$ does not exhibit and metric term did not participate with star product in Lagrangian density. additionally we construct the typical energy-momentum tensor for two cases of noncommutativity from general way and we show that these are symmetric tensors.

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